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Local orthogonal transform for a class of acoustic waveguides*

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Abstract There are some curved interfaces in acoustic waveguides. To compute wave propagation along the waveguides with some marching methods, flattening of the internal interfaces is needed. In this paper, a local orthogonal coordinate transform and an equation transform are constructed to solve the two-dimensional Helmholtz equation for the waveguides bounded by a flat top, a flat bottom and two curved internal interfaces with three layered media. The curved internal interfaces are flattened by the local orthogonal coordinate transform, and the corresponding transformed Helmholtz equation can be solved by some marching methods. This treatment can be extended in multilayered medium waveguides. The one-way reformulation based on the Dirichlet-to-Neumann (DtN) map is then used to reduce the boundary value problem to an initial value problem. Numerical implementation of the resulting operator Riccati equation uses a large range step method to discretize the range variable and a truncated local eigenfunction expansion to approximate the operators. This method is particularly useful for solving long range wave propagation problems in slowly varying waveguides with multilayered medium structure.

Keywords: **Helmholtz equation, local orthogonal transform, DtN reformulation, marching method, internal interface, one-way reformulation.**

The ocean environment is a well known acoustic waveguide with multilayered media and allows sound waves to travel a large distance in the horizontal direction. In this environment, the surface, in general, can be seen as even. The ocean bottom is composed of sediments and rocks. The interfaces between different layers are usually curved. The range distance L is many orders of magnitude larger than the typical wavelength. The depth D is much smaller than L, but still larger than the wavelength. This structure is very common in acoustics, electro-magnetism, seismic migration and other applications.

To compute wave propagation along the waveguides with some curved interfaces, a direct numerical computing is very expensive. Common numerical methods, such as the finite element method and the finite difference method, lead to very large linear systems. Meanwhile, these systems are also nonsymmetric and indefinite. Thus, it is very difficult to solve them by these methods. The coupled mode method[1], and some approximate methods[2—7] based on exact one-way reformulations are popularly efficient to solve the problem. However, these numerical studies focus on the waveguides with flat boundaries or interfaces. Although these methods can be used for the waveguide with curved boundary if the "staircase" approximation is used, it needs a small range step.

Numerical methods have been developed to avoid the crude "staircase" approximation. The approach of Refs. [8, 9] is to use a conformal mapping which keeps the governing equation in a very simple form. However, the conformal mapping is a global transformation that requires much effort for its calculation, especially when the waveguide is very long and the boundaries (or interfaces) are complicated. Local transformations are easier to compute and they have also been used in various applications[10—12]. Local but non-orthogonal transformations, such as the one used in Ref. [12], change the normal derivative to a combination of partial derivatives in the range and transverse variables. The range derivative at the interface or boundary can lead to difficulties in numerical implementation. Local orthogonal transform[13, 14] is an efficient method which flattens one curved bottom or one internal interface and changes the normal derivative at the interface only to the partial deriva-

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tive in the transverse variable. Furthermore, under an orthogonal transformation, the transformed Helmholtz equation does not involve the cross derivative term and it can be solved by some marching methods.

For the waveguides with at least two internal interfaces, it is necessary to use a transform that can flatten the curved interfaces. In this paper, we develop a new local orthogonal coordinate transform and a new equation transform for the acoustic waveguide with two curved internal interfaces and derive out the so-called "improved Helmholtz equation" with two flat interfaces. And some numerical examples are also given.

1 Basic equation

We start with the two-dimensional Helmholtz equation with two curved internal interfaces

$$u_{xx} + u_{zz} + \kappa^2(x,z)u = 0 \qquad (1)$$

for $-\infty < x < +\infty$, $0 < z < D_1$, where the first layer with density ρ_1 is located in $0 < z < h_1(x)$, the second layer with density ρ_2 is located in $h_1(x) < z < h_2(x)$, and the third layer with density ρ_3 is located in $h_2(x) < z < D_1$. The internal interfaces are two curves $z = h_1(x)$ and $z = h_2(x)$, where $D_1 > 1$, $L \gg D_1 \gg \frac{1}{k}$, u represents Fourier transform of acoustic pressure, and κ is called wavenumber. We also assume the problem is range independent(i.e. κ and h are independent of x) for $x \leqslant 0$ and $x \geqslant L$, that is

$$h_1(x) = \begin{cases} h_{1,0}, & x \leqslant 0 \\ h_{1,\infty}, & x \geqslant L \end{cases}$$

$$h_2(x) = \begin{cases} h_{2,0}, & x \leqslant 0 \\ h_{2,\infty}, & x \geqslant L \end{cases}$$

$$\kappa(x,z) = \begin{cases} \kappa_0(z), & x \leqslant 0 \\ \kappa_\infty(z), & x \geqslant L \end{cases}$$

The boundary conditions on the top and the bottom are supposed as $u|_{z=0} = 0$ and $u|_{z=D_1} = 0$. The interface conditions mean that

$$\begin{cases} \lim\limits_{z\to h_1(x)^-} u(x,z) = \lim\limits_{z\to h_1(x)^+} u(x,z), \\ \dfrac{1}{\rho_1} \lim\limits_{z\to h_1(x)^-} \dfrac{\partial u(x,z)}{\partial n} = \dfrac{1}{\rho_2} \lim\limits_{z\to h_1(x)^+} \dfrac{\partial u(x,z)}{\partial n}, \\ \lim\limits_{z\to h_2(x)^-} u(x,z) = \lim\limits_{z\to h_2(x)^+} u(x,z), \\ \dfrac{1}{\rho_2} \lim\limits_{z\to h_2(x)^-} \dfrac{\partial u(x,z)}{\partial n} = \dfrac{1}{\rho_3} \lim\limits_{z\to h_2(x)^+} \dfrac{\partial u(x,z)}{\partial n} \end{cases}$$

where n is a normal vector of the interface $z = h_1(x)$ or $z = h_2(x)$ (Fig. 1).

Since Helmholtz equation can be easily solved by separable variable method for $x \leqslant 0$ or $x \geqslant L$, we only need to solve the equation for $0 \leqslant x \leqslant L$. If there are no waves coming from $+\infty$, the exact boundary condition (radiation condition) at $x = L$ is $u_x = i\sqrt{\partial_z^2 + \kappa_\infty^2(z)}\,u$, where $i = \sqrt{-1}$ and the square root operator is defined in Ref. [4]. The simplest boundary condition at $x = 0$ is $u = u_0(z)$, where $u_0(z)$ is a given function of z.

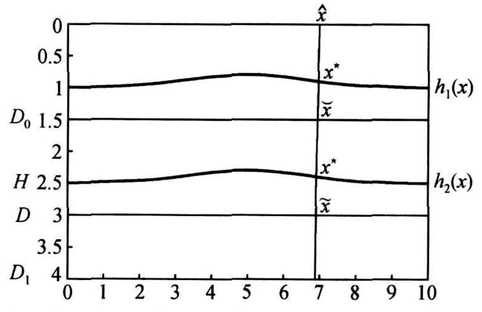

Fig. 1. Sketch map of waveguide with curved internal interfaces.

For simplicity of derivation, we develop our method for the waveguides with two curved internal interfaces. Here, we suppose that there is a straight line $z = D_0$ between $h_1(x)$ and $h_2(x)$, and there is a unique solution for Eq. (1) with the boundary conditions and interface conditions.

2 Local orthogonal transform

The acoustic waveguide is assumed as two parts separated by $z = D_0$. We transform the two parts into new coordinates $(\hat{x}, \hat{z})$. In the new coordinates, $h_1(x)$ is transformed into 1, $h_2(x)$ is transformed into H, and D_0 is transformed into D_0. To avoid squeezing the coordinate net in the new variable plane into narrow coordinate net in the (x, z) plane, we further divide the layer below $z = h_2(x)$ to two sublayers, $h_2(x) < z < D$ and $D < z < D_1$, with the added interface $z = D$. Then the marching methods can be applied in the new coordinates. The detailed transform scheme is as follows:

(i) The first layer $0 \leqslant z \leqslant h_1(x)$ in a medium with density[14] ρ_1. Let

$$\begin{cases} \hat{x} = f(x,z) \\ \hat{z} = g(x,z) = \dfrac{z}{h(x)} \end{cases}$$

satisfy

$$\{(x,z) \mid 0 \leqslant x \leqslant L, 0 \leqslant z \leqslant h_1(x)\} \xleftrightarrow{f,g} \{(\hat{x},\hat{z}) \mid 0 \leqslant \hat{x} \leqslant L, 0 \leqslant \hat{z} \leqslant 1\}$$

where the function f is to be determined. The transform is required to be orthogonal. Therefore,

$$\frac{\partial f}{\partial x} \cdot \frac{\partial g}{\partial x} + \frac{\partial f}{\partial z} \cdot \frac{\partial g}{\partial z} = 0$$

If $h_1'(x) \neq 0$, then the relationship between (x, z) and $(\hat{x}, \hat{z})$ is represented by

$$\int_{\hat{x}}^{x} \frac{h_1(t)}{h_1'(t)} \mathrm{d}t + \frac{1}{2}(\hat{z} h_1(x))^2 = 0$$

If $h_1'(x) = 0$, let $\hat{x} = x$.

(ii) The second layer $h_1(x) \leqslant z \leqslant D_0$ in a medium with the density[14] ρ_2. Let

$$\begin{cases} \hat{x} = f(x,z) \\ \hat{z} = g(x,z) = \dfrac{z - h_1(x)}{D_0 - h_1(x)} \cdot (D_0 - 1) + 1 \end{cases}$$

satisfy

$$\{(x,z) \mid 0 \leqslant x \leqslant L, h_1(x) \leqslant z \leqslant D_0\} \xleftrightarrow{f,g} \{(\hat{x},\hat{z}) \mid 0 \leqslant \hat{x} \leqslant L, 1 \leqslant \hat{z} \leqslant D_0\}$$

The relationship between (x, z) and $(\hat{x}, \hat{z})$ is represented by

$$\int_{x^*}^{x} \frac{D_0 - h_1(t)}{h_1'(t)} \mathrm{d}t - \frac{1}{2}[(z - D_0)^2 - (h_1(x^*) - D_0)^2] = 0$$

where $(x^*, h_1(x^*)) \xleftrightarrow{1\text{-}1} (\hat{x}, 1)$. If $h_1'(x) = 0$, let $\hat{x} = x$.

(iii) The third layer $D_0 \leqslant z \leqslant h_2(x)$ in a medium with the density ρ_2. Let $\hat{x} = f(x, z)$ and

$$\begin{cases} \hat{x} = f(x,z) \\ \hat{z} = g(x,z) = \dfrac{z - D_0}{h_2(x) - D_0} \cdot (H - D_0) + D_0 \end{cases}$$

satisfy

$$\{(x,z) \mid 0 \leqslant x \leqslant L, D_0 \leqslant z \leqslant h_2(x)\} \xleftrightarrow{f,g} \{(\hat{x},\hat{z}) \mid 0 \leqslant \hat{x} \leqslant L, D_0 \leqslant \hat{z} \leqslant H\}$$

The relationship between (x, z) and $(\hat{x}, \hat{z})$ is represented by

$$\int_{\breve{x}}^{x} \frac{h_2(t) - D_0}{h_2'(t)} \mathrm{d}t + \frac{1}{2}(z - D_0)^2 = 0$$

where $(\breve{x}, h_2(\breve{x})) \xleftrightarrow{1\text{-}1} (\hat{x}, D_0)$. If $h_2'(x) = 0$, let $\breve{x} = x$.

(iv) The fourth layer $h_2(x) \leqslant z \leqslant D$ in a medium with the density ρ_3. Let

$$\begin{cases} \hat{x} = f(x,z) \\ \hat{z} = g(x,z) = \dfrac{z - h_2(x)}{D - h_2(x)} \cdot (D - H) + H \end{cases}$$

satisfy

$$\{(x,z) \mid 0 \leqslant x \leqslant L, h_2(x) \leqslant z \leqslant D\} \xleftrightarrow{f,g} \{(\hat{x},\hat{z}) \mid 0 \leqslant \hat{x} \leqslant L, H \leqslant \hat{z} \leqslant D\}$$

The relationship between (x, z) and $(\hat{x}, \hat{z})$ is represented by

$$\int_{x^*}^{x} \frac{D - h_2(t)}{h_2'(t)} \mathrm{d}t - \frac{1}{2}[(z - D)^2 - (h_2(x^*) - D)^2] = 0$$

where $(x^*, h_2(x^*)) \xleftrightarrow{1\text{-}1} (\hat{x}, H)$ and $(\tilde{x}, D) \xleftrightarrow{1\text{-}1} (\hat{x}, D)$. If $h_2'(x) = 0$, let $\breve{x} = x$.

(v) The fifth layer $D \leqslant \hat{z} \leqslant D_1$ in a medium with the density ρ_3. Let

$$\begin{cases} \hat{x} = f(x,D) \\ \hat{z} = g(x,z) = z \end{cases}$$

satisfy

$$\{(x,z) \mid 0 \leqslant x \leqslant L, D \leqslant z \leqslant D_1\} \xleftrightarrow{f,g} \{(\hat{x},\hat{z}) \mid 0 \leqslant \hat{x} \leqslant L, D \leqslant \hat{z} \leqslant D_1\}$$

3 Equation transformation

Because Eq. (1) is expected to be transformed as

$$V_{\hat{x}\hat{x}} + \alpha(\hat{x},\hat{z}) V_{\hat{z}\hat{z}} + \beta(\hat{x},\hat{z}) V_{\hat{z}} + \gamma(\hat{x},\hat{z}) V = 0 \tag{2}$$

we let $u(x,z) = W(x,z) \cdot V(x,z)$, where W can be derived out by the idea in Ref. [14]. The coefficients of Eq. (2) are obtained as follows:

$$\begin{cases} \alpha(\hat{x},\hat{z}) = \dfrac{g_z^2 + g_x^2}{f_z^2 + f_x^2} \\ \beta(\hat{x},\hat{z}) = \dfrac{2W_z g_z + W g_{zz} + 2W_x g_x + W g_{xx}}{W(f_z^2 + f_x^2)} \\ \gamma(\hat{x},\hat{z}) = \dfrac{W_{xx} + W_{zz} + \kappa^2 W}{W(f_z^2 + f_x^2)} \end{cases} \tag{3}$$

In this section, we will develop an efficient algorithm to compute the coefficients of transformed equation in the multilayered waveguide. We have

$$W(x,z)=\begin{cases}\sqrt{\dfrac{h_1'(x)\cdot h_1(\hat{x})}{h_1'(\hat{x})\cdot h_1^2(x)}}, & 0\leqslant z\leqslant h_1(x)\\ \sqrt{\dfrac{h_1(\hat{x})}{h_1'(\hat{x})}\cdot\dfrac{D_0-h_1(x^*)}{h_1(x^*)}\cdot\dfrac{h_1'(x)}{[D_0-h_1(x)]^2}}, & h_1(x)\leqslant z\leqslant D_0\\ P(\breve{x})\sqrt{\dfrac{h_2'(x)\cdot(h_2(\breve{x})-D_0)}{h_2'(\breve{x})\cdot(h_2(x)-D_0)^2}}, & D_0\leqslant z\leqslant h_2(x)\\ P(\breve{x})\sqrt{\dfrac{h_2(\breve{x})-D_0}{h_2'(\breve{x})}\cdot\dfrac{D-h_2(x^*)}{h_2(x^*)-D_0}\cdot\dfrac{h_2'(x)}{[D-h_2(x)]^2}}, & h_2(x)\leqslant z\leqslant D\\ P(\breve{x})\sqrt{\dfrac{h_2(\breve{x})-D_0}{h_2'(\breve{x})}\cdot\dfrac{D-h_2(x^*)}{h_2(x^*)-D_0}\cdot\dfrac{h_2'(x)}{D-h_2(x)}}, & D\leqslant z\leqslant D_1\end{cases}\tag{4}$$

$$f_x(x,z)=\begin{cases}\dfrac{h_1'(\hat{x})h_1(x)}{h_1'(x)h_1(\hat{x})}, & 0\leqslant z\leqslant h_1(x)\\ \dfrac{h_1'(\hat{x})[D_0-h_1(x)]h_1(x^*)}{h_1(\hat{x})h_1'(x)[D_0-h_1(x^*)]}, & h_1(x)\leqslant z\leqslant D_0\\ P(\breve{x})\dfrac{h_2'(\breve{x})[h_2(x)-D_0]}{h_2'(x)[h_2(\breve{x})-D_0]}, & D_0\leqslant z\leqslant h_2(x)\\ P(\breve{x})\dfrac{h_2'(\breve{x})[D-h_2(x)][h_2(x^*)-D_0]}{[h_2(\breve{x})-D_0]h_2'(x)[D-h_2(x^*)]}, & h_2(x)\leqslant z\leqslant D\\ P(\breve{x})\dfrac{h_2'(\breve{x})[D-h_2(\tilde{x})][h_2(x^*)-D_0]}{[h_2(\breve{x})-D_0]h_2'(\tilde{x})[D-h_2(x^*)]} & D\leqslant z\leqslant D_1\end{cases}\tag{5}$$

$$f_z(x,z)=\begin{cases}z\dfrac{h_1'(\hat{x})}{h_1(\hat{x})}, & 0\leqslant z\leqslant h_1(x)\\ \dfrac{h_1(x^*)h_1'(\hat{x})[D_0-z]}{h_1(\hat{x})[D_0-h_1(x^*)]}, & h_1(x)\leqslant z\leqslant D_0\\ P(\breve{x})[z-D_0]\dfrac{h_2'(\breve{x})}{h_2(\breve{x})-D_0}, & D_0\leqslant z\leqslant h_2(x)\\ P(\breve{x})\dfrac{[h_2(x^*)-D_0]h_2'(\breve{x})[D-z]}{[h_2(\breve{x})-D_0][D-h_2(x^*)]}, & h_2(x)\leqslant z\leqslant D\\ 0 & D\leqslant z\leqslant D_1\end{cases}\tag{6}$$

$$g_x(x,z)=\begin{cases}\dfrac{-zh_1'(x)}{h_1^2(x)}, & 0\leqslant z\leqslant h_1(x)\\ \dfrac{[z-D_0]h_1'(x)[D_0-1]}{[D_0-h_1(x)]^2}, & h_1(x)\leqslant z\leqslant D_0\\ \dfrac{-[z-D_0]h_2'(x)[H-D_0]}{[h_2(x)-D_0]^2}, & D_0\leqslant z\leqslant h_2(x)\\ \dfrac{[z-D]h_2'(x)[D-H]}{[D-h_2(x)]^2}, & h_2(x)\leqslant z\leqslant D\\ 0 & D\leqslant z\leqslant D_1\end{cases}\tag{7}$$

and

$$g_z(x,z)=\begin{cases}\dfrac{1}{h_1(x)}, & 0\leqslant z\leqslant h_1(x)\\ \dfrac{D_0-1}{D_0-h_1(x)}, & h_1(x)\leqslant z\leqslant D_0\\ \dfrac{H-D_0}{h_2(x)-D_0}, & D_0\leqslant z\leqslant h_2(x)\\ \dfrac{D-H}{D-h_2(x)}, & h_2(x)\leqslant z\leqslant D\\ 1 & D\leqslant z\leqslant D_1\end{cases} \tag{8}$$

where

$$P(\breve{x})=\left[\frac{h_1'(\breve{x})}{h_1'(\hat{x})}\right]^{\frac{1}{2}}\left[\frac{D_0-h_1(x^*)}{D_0-h_1(\breve{x})}\right]^{\frac{1}{2}}\left[\frac{h_1(\hat{x})}{h_1(x^*)}\right]^{\frac{1}{2}}$$

Here, the waveguide is divided into two parts (Part I, Part II) by $z=D_0$ between the two curved interfaces $z=h_1(x)$ and $z=h_2(x)$.

In Part I, the coefficients are the same as that obtained in Ref. [14], and it is expressed as a special case of $\alpha_j(\hat{x},\hat{z}),\beta_j(\hat{x},\hat{z}),\gamma_j(\hat{x},\hat{z})(j=1,2)$ listed in Appendix 1, where $h(x),x^*,D_0,H$, and D are substituted by $h_1(x),x^*,0,1$, and D_0, respectively. In the first layer,

$$\begin{cases}\alpha(\hat{x},\hat{z})=\alpha_1(\hat{x},\hat{z})\\ \beta(\hat{x},\hat{z})=\beta_1(\hat{x},\hat{z})\\ \gamma(\hat{x},\hat{z})=\gamma_1(\hat{x},\hat{z})\end{cases} \tag{9}$$

In the second layer,

$$\begin{cases}\alpha(\hat{x},\hat{z})=\alpha_2(\hat{x},\hat{z})\\ \beta(\hat{x},\hat{z})=\beta_2(\hat{x},\hat{z})\\ \gamma(\hat{x},\hat{z})=\gamma_2(\hat{x},\hat{z})\end{cases} \tag{10}$$

In Part II, the coefficients are expressed as formulas constructed from $\alpha_j(\hat{x},\hat{z}),\beta_j(\hat{x},\hat{z}),\gamma_j(\hat{x},\hat{z})$ $(j=1,2,3)$ listed in Appendix 1, where $h(x)$ and $\hat{x}$ are substituted by $h_2(x)$ and $\breve{x}$, respectively. In the third layer,

$$\begin{cases}\alpha(\hat{x},\hat{z})=M(\hat{x})\alpha_1(\hat{x},\hat{z})\\ \beta(\hat{x},\hat{z})=M(\hat{x})\beta_1(\hat{x},\hat{z})\\ \gamma(\hat{x},\hat{z})=M(\hat{x})\gamma_1(\hat{x},\hat{z})+M(\hat{x})\dfrac{P_{\breve{x}\breve{x}}}{P}\end{cases} \tag{11}$$

In the fourth layer,

$$\begin{cases}\alpha(\hat{x},\hat{z})=M(\hat{x})\alpha_2(\hat{x},\hat{z})\\ \beta(\hat{x},\hat{z})=M(\hat{x})\beta_2(\hat{x},\hat{z})\\ \gamma(\hat{x},\hat{z})=M(\hat{x})\gamma_2(\hat{x},\hat{z})+M(\hat{x})\dfrac{P_{\breve{x}\breve{x}}}{P}\end{cases} \tag{12}$$

In the fifth layer,

$$\begin{cases}\alpha(\hat{x},\hat{z})=M(\hat{x})\alpha_3(\hat{x},\hat{z})\\ \beta(\hat{x},\hat{z})=M(\hat{x})\beta_3(\hat{x},\hat{z})\\ \gamma(\hat{x},\hat{z})=M(\hat{x})\gamma_3(\hat{x},\hat{z})+M(\hat{x})\dfrac{P_{\breve{x}\breve{x}}}{P}\end{cases} \tag{13}$$

where $M(\hat{x})=P(\breve{x})^4$, $M(\hat{x})\dfrac{P_{\breve{x}\breve{x}}}{P}$ is given in Appendix 2.

Remark 1. The formulas (9)—(13) can degenerate into the previous formulas with one internal interface when $h_1(x)$ or $h_2(x)$ becomes straight line.

Remark 2. Formulas for coefficients of transformed system in an arbitrarily layered medium structure can be developed by extending (9)—(13).

4 Interface conditions

Range discretization and matrix approximations are the same as that in Ref. [14], so we omit them here for simplicity. We only list the differences for the boundary and interface conditions.

For Eq. (2), the top and bottom boundary conditions are

$$V|_{\hat{z}=0}=0,\quad V|_{\hat{z}=D_1}=0$$

The interface conditions between the first and the second layer (at $\hat{z}=1$) become

$$(WV)|_{\hat{z}=1^-}=(WV)|_{\hat{z}=1^+}$$

$$\frac{1}{\rho_1}W\left\{\frac{1}{2}\left[h_1''(x)-2\frac{\{h_1'(x)\}^2}{h_1(x)}\right]V-\frac{1+\{h_1'(x)\}^2}{h_1(x)}V_{\hat{z}}\right\}\Bigg|_{\hat{z}\to1^-}$$
$$=\frac{1}{\rho_2}W\left\{\frac{1}{2}\left[h_1''(x)+2\frac{\{h_1'(x)\}^2}{D_0-h_1(x)}\right]V-\frac{D_0-1}{D_0-h_1(x)}[1+\{h_1'(x)\}^2]V_{\hat{z}}\right\}\Bigg|_{\hat{z}\to1^+}$$

At the interface $\hat{z}=D_0$ between the second and the third layer,

$$WV|_{\hat{z}\to D_0^-}=WV|_{\hat{z}\to D_0^+},$$

$$W\frac{1-D_0}{h_1(x)-D_0}V_{\hat{z}}\Bigg|_{\hat{z}\to D_0^-}=W\frac{H-D_0}{h_2(x)-D_0}V_{\hat{z}}\Bigg|_{\hat{z}\to D_0^+}$$

The interface conditions between the third and the fourth layer (at $\hat{z}=H$) become

$$\frac{1}{\rho_2}\frac{W}{P(\breve{x})}\left\{\left\{\left[h_2'(x)\frac{d\breve{x}}{dx}-\frac{d\breve{x}}{dz}\right]P_{\breve{x}}+\frac{1}{2}P(\breve{x})\left[h_2''(x)-2\frac{\{h_2'(x)\}^2}{h_2(x)}\right]\right\}V-P(\breve{x})\frac{1+\{h_2'(x)\}^2}{h_2(x)}V_{\hat{z}}\right\}\Bigg|_{\hat{z}\to H^-}$$

$$=\frac{1}{\rho_3}\frac{W}{P(\breve{x})}\left\{\left\{\left[h_2'(x)\frac{d\breve{x}}{dx}-\frac{d\breve{x}}{dz}\right]P_{\breve{x}}+\frac{1}{2}P(\breve{x})\left[h_2''(x)+2\frac{\{h_2'(x)\}^2}{D-h_2(x)}\right]\right\}V-P(\breve{x})\frac{D-H}{D-h_2(x)}[1+\{h_2'(x)\}^2]V_{\hat{z}}\right\}\Bigg|_{\hat{z}\to H^+}$$

where

$$\begin{cases}\dfrac{d\breve{x}}{dx}=\dfrac{h_2'(\breve{x})[h_2(x)-D_0]}{h_2'(x)[h_2(\breve{x})-D_0]},\ \dfrac{d\breve{x}}{dz}=\dfrac{h_2'(\breve{x})}{[h_2(\breve{x})-D_0]}[\hat{z}-D_0] & D_0<\hat{z}<H\\ \dfrac{d\breve{x}}{dx}=\dfrac{[h_2(x^*)-D_0]h_2'(\breve{x})[D-h_2(x)]}{[h_2(\breve{x})-D_0]h_2'(x)[D-h_2(x^*)]},\ \dfrac{d\breve{x}}{dz}=\dfrac{[h_2(x^*)-D_0]h_2'(\breve{x})[D-h_2(x)]}{[h_2(\breve{x})-D_0][D-h_2(x^*)]}, & H<\hat{z}<D\end{cases}$$

and

$$P_{\breve{x}}=\left\{h_1'(\hat{x})^2-h_1''(\hat{x})h_1(\hat{x})+\frac{h_1'(\breve{x})^2h_1(\hat{x})^2[D_0-h_1(x^*)]}{[D_0-h_1(\breve{x})]^2h_1(x^*)}+\frac{h_1''(\breve{x})^2h_1(\hat{x})^2[D_0-h_1(x^*)]}{[D_0-h_1(\breve{x})]h_1(x^*)}+\frac{D_0h_1'(x^*)^2h_1(\hat{x})}{[1+h_1'(x^*)^2][D_0-h_1(x^*)]h_1(x^*)}\right\}\Bigg/\left\{2h_1(\hat{x})^2[h_1'(\hat{x})h_1'(\breve{x})]^{\frac{1}{2}}\cdot\left[\frac{h_1(\hat{x})}{D_0-h_1(\breve{x})}\right]^{\frac{1}{2}}\left[\frac{D_0-h_1(x^*)}{h_1(x^*)}\right]^{\frac{1}{2}}\right\}$$

When $h_1'(x)=0$ or $h_2'(x)=0$, the formulas can be obtained by referencing Appendixes 1 and 2. Details are omitted here.

At the interface between the fourth and the fifth layer (at $\hat{z}=D$),

$$WV\,|_{\hat{z}\to D^-}=WV\,|_{\hat{z}\to D^+},$$

$$W\frac{H-D}{h_2(x)-D}V_{\hat{z}}\Bigg|_{\hat{z}\to D^-}=WV_{\hat{z}}\,|_{\hat{z}\to D^+}$$

5 Numerical examples

The method presented in the previous section has been tested on a number of examples. Four of them are given below. In the first example, the waveguide is divided into three layers by two curved interfaces $z=h_1(x)$ and $z=h_2(x)$, and the peaks of $h_1(x)$ and $h_2(x)$ are at the same x-position; the second example is the example in Ref. [14], we redo it by our method; the third one is with two curved interfaces $z=h_1(x)$, $z=h_2(x)$ with peaks at different x-positions; and the fourth one considers a thin strip between two curved internal interfaces.

Example 1. Let

$$\kappa=\begin{cases}16, & 0<z<h_1(x)\\ 0.7\times16, & h_1(x)<z<h_2(x)\\ 0.2\times16, & h_2(x)<z<D_1\end{cases}$$

with $L=10$, $n=30$, $D_0=1.5$, $H=2.5$, $D=3$, $D_1=4$, $N=400$, $\rho_1=1$, $\rho_2=1.7$, $\rho_3=2.7$, $h_1(x)=1-\varepsilon_1 e^{-\sigma_1\left(\frac{x}{L}-\frac{1}{2}\right)^2}$, $h_2(x)=H-\varepsilon_2 e^{-\sigma_2\left(\frac{x}{L}-\frac{1}{2}\right)^2}$, $\varepsilon_1=\varepsilon_2=0.1$, $\sigma_1=\sigma_2=10$, $0\leqslant z\leqslant 4$, and $0\leqslant x\leqslant 10$, where N is the number of points to discretize the $\hat{z}$ variable, n is the number to truncate the $N\times N$ matrices that approximate the operators appearing in marching process.[14]

Here, $V(0,\hat{z})$ is given by the eigenfunction whose corresponding eigenvalue is the largest one at $x=L$. After flattening the two curved interfaces by the orthogonal transform we suggest, we can use the marching scheme[14] to compute the solution $u(x,z)$ at $x=L$. The corresponding solution is shown in Fig. 2.

Example 2. Let

$$\kappa=\begin{cases}16, & 0<z<h_1(x)\\ 0.7\times16, & h_1(x)<z<h_2(x)\\ 0.7\times16, & h_2(x)<z<D_1\end{cases}$$

with $L=10$, $n=30$, $D_0=1.5$, $H=2.5$, $D=3$, $D_1=4$, $N=400$, $\rho_1=1$, $\rho_2=1.7$, $\rho_3=1.7$, $h_1(x)=1-\varepsilon_1 e^{-\sigma_1\left(\frac{x}{L}-\frac{1}{2}\right)^2}$, $h_2(x)=H$, $\varepsilon_1=0.2$, $\sigma_1=20$, $0\leqslant z\leqslant 4$, and $0\leqslant x\leqslant 10$.

The corresponding solution is shown in Fig. 3.

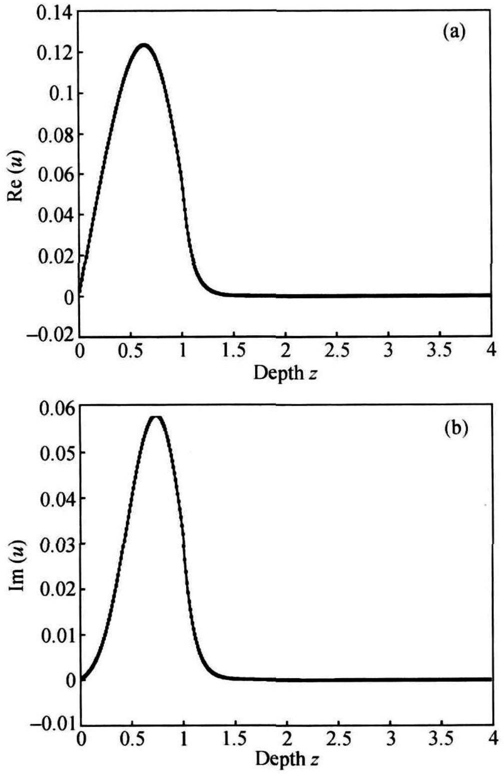

Fig. 2. Comparison of $u(L,z)$ for $\tau=\frac{1}{4}$ and $\tau=\frac{1}{256}$ in Example 1. (a) The real part; (b) the imaginary part.

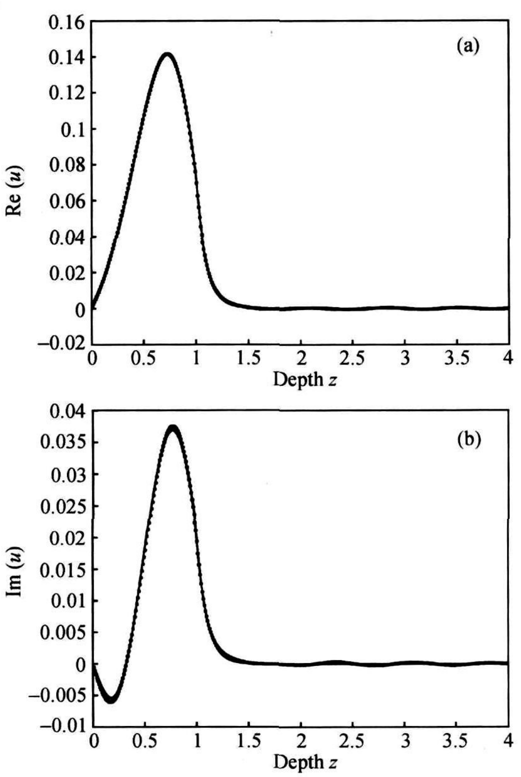

Fig. 3. Comparison of $u(L,z)$ for $\tau=\frac{1}{4}$ and $\tau=\frac{1}{256}$ in Example 2. (a) The real part; (b) the imaginary part.

Example 3. Let

$$\kappa=\begin{cases}16, & 0<z<h_1(x)\\ 0.7\times 16, & h_1(x)<z<h_2(x)\\ 0.2\times 16, & h_2(x)<z<D_1\end{cases}$$

with $L=10$, $n=30$, $D_0=1.5$, $H=2.5$, $D=3$, $D_1=4$, $N=400$, $\rho_1=1$, $\rho_2=1.7$, $\rho_3=2.7$, $h_1(x)=1-\varepsilon_1 e^{-\sigma_1\left(\frac{x}{L}-\frac{1}{4}\right)^2}$, $h_2(x)=H-\varepsilon_2 e^{-\sigma_2\left(\frac{x}{L}-\frac{3}{4}\right)^2}$, $\varepsilon_1=\varepsilon_2=0.1$, $\sigma_1=\sigma_2=20$, $0\leqslant z\leqslant 4$, and $0\leqslant x\leqslant 10$.

The corresponding solution is shown in Fig. 4.

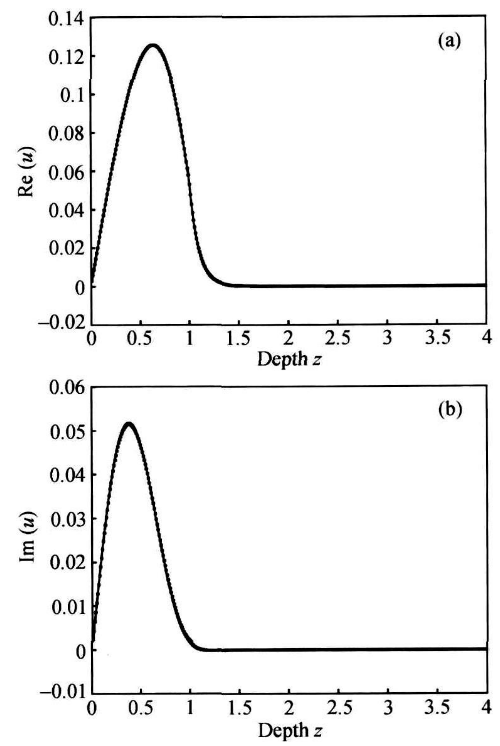

Fig. 4. Comparison of $u(L,z)$ for $\tau=\frac{1}{4}$ and $\tau=\frac{1}{256}$ in Example 3. (a) The real part; (b) the imaginary part.

Example 4. Let

$$\kappa=\begin{cases}16, & 0<z<h_1(x)\\ 0.7\times 16, & h_1(x)<z<h_2(x)\\ 0.2\times 16, & h_2(x)<z<D_1\end{cases}$$

with $L=10$, $n=30$, $D_0=1.1$, $H=1.2$, $D=3$, $D_1=4$, $N=400$, $\rho_1=1$, $\rho_2=1.7$, $\rho_3=2.7$, $h_1(x)=1-\varepsilon_1 e^{-\sigma_1\left(\frac{x}{L}-\frac{1}{2}\right)^2}$, $h_2(x)=H-\varepsilon_2 e^{-\sigma_2\left(\frac{x}{L}-\frac{1}{2}\right)^2}$, $\varepsilon_1=0.1$, $\varepsilon_2=0.05$, $\sigma_1=\sigma_2=20$, $0\leqslant z\leqslant 4$, and $0\leqslant x\leqslant 10$.

The corresponding solution is shown in Fig. 5.

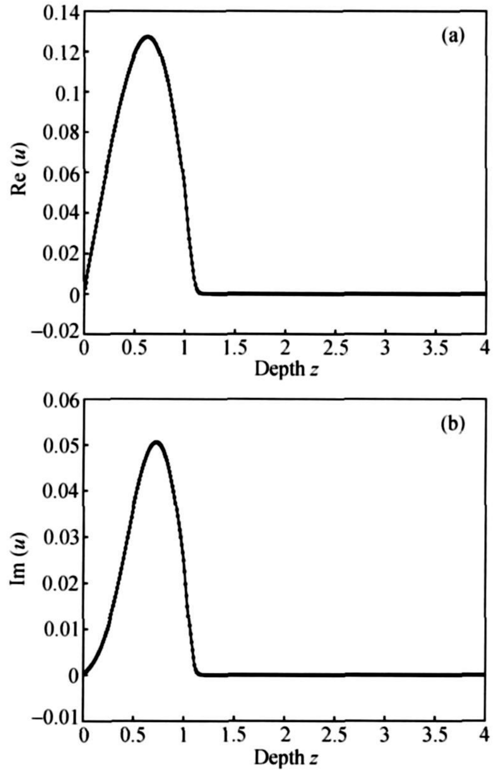

Fig. 5. Comparison of $u(L, z)$ for $\tau = \frac{1}{4}$ and $\tau = \frac{1}{256}$ in Example 4. (a) The real part; (b) the imaginary part.

In Figs. 2—5, the solid line is represented as solution for $\tau = \frac{1}{256}$, and bold points are represented as the solution for $\tau = \frac{1}{4}$. The solution is obtained with $\tau = \frac{1}{256}$ as the "exact" solution, then we calculate the relative errors of $u(L, z)$. The relative errors of Examples 1—4 are 0.0016, 0.0102, 0.0041 and 0.0023, respectively. The numerical examples demonstrate that more accurate approximate solutions can be obtained by quite large steps.

6 Conclusion

The result of this work has provided a theoretical foundation for developing practical numerical scheme for a class of acoustic waveguides with three layered media and two internal curved interfaces. By constructing local orthogonal coordinate transformations, the interfaces are flattened. This extends the previous work[14] which covers waveguides with one curved internal interface. Furthermore, this treatment can be applied to waveguides with more complex interfaces. This method is particularly useful for solving long range wave propagation problems in slowly varying waveguides with multilayered medium structure. Numerical examples demonstrate that our method is feasible for solving the Helmholtz equation with three layer media by a large range step. Since the derivatives of interface function are used to derive the transform, the method cannot be applied to the case where the interface is piecewise smooth (with corners), unless the piecewise smooth interface is first approximated by a smooth one. In addition, we will study for the case that there is not a straight line between two interfaces in future.

Appendix 1

When $h'(\bar{x}) = 0$, we define $\alpha(\bar{x}, \hat{z}) = \lim_{x \to \bar{x}} \alpha(\hat{x}, \hat{z})$, $\beta(\bar{x}, \hat{z}) = \lim_{x \to \bar{x}} \beta(\hat{x}, \hat{z})$ and $\gamma(\bar{x}, \hat{z}) = \lim_{x \to \bar{x}} \gamma(\hat{x}, \hat{z})$. The formulas for coefficients α_j, β_j, γ_j, $j = 1, 2, 3$ are given as follows:

If $h'(x) \neq 0$, then

$$\begin{cases} \alpha_1(\hat{x}, \hat{z}) = \dfrac{[H - D_0]^2 [h'(x)]^2 [h(\hat{x}) - D_0]^2}{[h(x) - D_0]^4 [h'(\hat{x})]^2} \\ \beta_1(\hat{x}, \hat{z}) = \dfrac{2(z - D_0)[H - D_0]\{2[h'(x)]^2 - [h(x) - D_0]h''(x)\}[h'(x)]^2 [h(\hat{x}) - D_0]^2}{[h(x) - D_0]^3 [h'(\hat{x})]^2 \{[h(x) - D_0]^2 + (z - D_0)^2 [h'(x)]^2\}} \\ \gamma_1(\hat{x}, \hat{z}) = \Bigg\{ 2\dfrac{[h'(x)]^2}{[h(x) - D_0]^2} - 2\dfrac{h''(x)}{h(x) - D_0} + \dfrac{(z - D_0)^2}{4}\Bigg[\dfrac{\{h''(\hat{x})\}^2 - 2h'(\hat{x})h'''(\hat{x})}{[h(\hat{x}) - D_0]^2} \\ \qquad + \dfrac{4h'(\hat{x})h''(\hat{x})}{[h(\hat{x}) - D_0]^3} - 3\dfrac{\{h'(\hat{x})\}^4}{[h(\hat{x}) - D_0]^4} \Bigg] + \dfrac{[h(x) - D_0]^2}{4[h'(x)]^2}\Bigg[\dfrac{2h'(x)h'''(x) - \{h''(x)\}^2}{[h(x) - D_0]^2} \\ \qquad - \dfrac{2h'(\hat{x})h'''(\hat{x}) - \{h''(\hat{x})\}^2}{[h(\hat{x}) - D_0]^2} \Bigg] + \dfrac{[h'(\hat{x})]^2}{[h'(x)]^2}\Bigg[\dfrac{[h(x) - D_0]^2 h''(\hat{x})}{[h(\hat{x}) - D_0]^3} \\ \qquad - \dfrac{3}{4}\dfrac{[h(x) - D_0]^2 [h'(\hat{x})]^2}{[h(\hat{x}) - D_0]^4} \Bigg] + \kappa^2(x, z) \Bigg\} \Bigg/ (f_x^2 + f_z^2) \end{cases}$$

where

$$f_x = \frac{h'(\hat{x})[h(x) - D_0]}{h'(x)[h(\hat{x}) - D_0]},$$

$$f_z = [z - D_0]\frac{h'(\hat{x})}{h(\hat{x}) - D_0}$$

If $h'(\bar{x})=0$ and $h''(\bar{x})\neq 0$, then

$$\begin{cases} \alpha_1(\bar{x},\hat{z}) = \dfrac{[H-D_0]^2}{[h(\bar{x})-D_0]^2} e^{-(z-D_0)^2 \frac{h''(\bar{x})}{[h(\bar{x})-D_0]}} \\ \beta_1(\bar{x},\hat{z}) = (-2)(z-D_0)(H-D_0) \cdot \dfrac{h''(\bar{x})}{[h(\bar{x})-D_0]^2} e^{-(z-D_0)^2 \frac{h''(\bar{x})}{[h(\bar{x})-D_0]}} \\ \gamma_1(\bar{x},\hat{z}) = \left\{ \dfrac{h''(\bar{x})}{[h(\bar{x})-D_0]} [e^{(z-D_0)^2 \frac{h''(\bar{x})}{[h(\bar{x})-D_0]}} - 2] + \dfrac{[h''(\bar{x})]^2 + [h(\bar{x})-D_0]h^{(4)}(\bar{x})}{4[h(\bar{x})-D_0]h''(\bar{x})} \cdot [1 - e^{(z-D_0)^2 \frac{h''(\bar{x})}{[h(\bar{x})-D_0]}}] + \dfrac{(z-D_0)^2}{4} \cdot \dfrac{[h''(\bar{x})]^2}{[h(\bar{x})-D_0]^2} + \kappa^2(\bar{x},z) \right\} e^{-(z-D_0)^2 \frac{h''(\bar{x})}{[h(\bar{x})-D_0]}} \end{cases}$$

with $\lim\limits_{\hat{x}\to\bar{x}} \dfrac{h'(x)}{h'(\hat{x})} = e^{-\frac{(z-D_0)^2}{2}\frac{h''(\bar{x})}{[h(\bar{x})-D_0]}}$. If $h'(\bar{x})=0$ and $h''(\bar{x})=0$, then $\alpha_1(\bar{x},\hat{z}) = \dfrac{1}{[h(\bar{x})-D_0]^2}$, $\beta_1(\bar{x},\hat{z})=0$, and

$$\gamma_1(\bar{x},\hat{z}) = \kappa^2(\bar{x},z) - \frac{(z-D_0)^2}{4} \cdot \frac{h^{(4)}(\bar{x})}{[h(\bar{x})-D_0]}$$

If $h'(x)\neq 0$, then

$$\begin{cases} \alpha_2(\hat{x},\hat{z}) = (D-H)^2 \cdot \dfrac{[D-h(x^*)]^2[h'(x)]^2[h(\hat{x})-D_0]^2}{[D-h(x)]^4[h'(\hat{x})]^2[h(x^*)-D_0]^2} \\ \beta_2(\hat{x},\hat{z}) = 2(D-H)(z-D) \times \dfrac{2[h'(x)]^2 + [D-D_0]h''(x) - h''(x)h'(x)}{[D-h(x)]^2 + [h'(x)]^2(D-z)^2} \times \left[\dfrac{h'(x)}{h'(\hat{x})}\right]^2 \left[\dfrac{h(\hat{x})-D_0}{h(x^*)}\right]^2 \dfrac{[D-h(x^*)]^2}{[D-h(x)]^3} \\ \gamma_2(\hat{x},\hat{z}) = \left\{ \dfrac{\left[\dfrac{D-h(x)}{h'(x)}\right]^2 + (D-z)^2}{[D-h(x^*)]^2} \cdot \left[\dfrac{[h(x^*)-D_0]^2}{[h(\hat{x})-D_0]^2} \left(-\dfrac{3}{4}\dfrac{[h'(\hat{x})]^4}{[h(\hat{x})-D_0]^2} + \dfrac{[h'(\hat{x})]^4 h''(\hat{x})}{h(\hat{x})-D_0} + \dfrac{1}{4}[h''(\hat{x})]^2 - \dfrac{1}{2}h'(\hat{x})h'''(\hat{x}) \right) + \dfrac{[D-D_0][h'(x^*)]^2}{[h(x^*)-D_0][D-h(x^*)][1+\{h'(x^*)\}^2]^2} \times \left(\dfrac{3}{4}\dfrac{[h'(x^*)]^2\{D-2h(x^*)+D_0\}}{[h(x^*)-D_0]\{D-h(x^*)\}} - \dfrac{h''(x^*)}{1+\{h'(x^*)\}^2} \right) \right] + 2\dfrac{[h'(x)]^2}{[D-h(x)]^2} + 2\dfrac{h''(x)}{D-h(x)} + \dfrac{1}{2}\dfrac{h'''(x)}{h'(x)} - \dfrac{1}{4}\dfrac{[h''(x)]^2}{[h'(x)]^2} + \kappa^2(x,z) \right\} \cdot \dfrac{[h(\hat{x})-D_0]^2[D-h(x^*)]^2[h'(x)]^2}{[h(x^*)-D_0]^2[D-h(x)]^2[h'(\hat{x})]^2} \end{cases}$$

If $h'(\bar{x})=0$ and $h''(\bar{x})\neq 0$, then

$$\begin{cases} \alpha_2(\bar{x},\hat{z}) = \dfrac{(D-H)^2}{[D-h(\bar{x})]^2} \cdot e^{\frac{h''(\bar{x})}{D-h(\bar{x})}\{(D-D_0)\cdot[h(\bar{x})-D_0] - 2(D-D_0)\cdot(z-D_0) + (z-D_0)^2\}} \\ \beta_2(\bar{x},\hat{z}) = \dfrac{2(D-H)(z-D)h''(\bar{x})}{[D-h(\bar{x})]^3} \cdot e^{\frac{h''(\bar{x})}{D-h(\bar{x})}\{(z-D_0)^2 - 2(D-D_0)\cdot(z-D_0) + (D-D_0)\cdot[h(\bar{x})-D_0]\}} \\ \gamma_2(\bar{x},\hat{z}) = e^{\frac{h''(\bar{x})}{D-h(\bar{x})}\{(z-D_0)^2 - 2(D-D_0)\cdot(z-D_0) + (D-D_0)\cdot[h(\bar{x})-D_0]\}} \cdot \left\{ \dfrac{h''(\bar{x})}{h(\bar{x})-D_0} \left[\dfrac{[h(\bar{x})-D_0]h''(\bar{x})(D-z)^2}{\{D-h(\bar{x})\}^2} - \dfrac{[h(\bar{x})-D_0]}{D-h(\bar{x})} + 3e^{-\frac{h''(\bar{x})}{D-h(\bar{x})}\{(z-D_0)^2 - 2(D-D_0)\cdot(z-D_0) + (D-D_0)\cdot[h(\bar{x})-D_0]\}} \right] - \dfrac{3}{4} \cdot \dfrac{(D-D_0)\cdot h''(\bar{x})}{[h(\bar{x})-D_0][D-h(\bar{x})]} \cdot e^{\frac{h''(\bar{x})}{D-h(\bar{x})}(z-h(\bar{x}))(2D-z-h(\bar{x}))} + \dfrac{2h''(\bar{x})}{D-h(\bar{x})} + \kappa^2(\bar{x},z) + \dfrac{h^{(4)}(\bar{x})}{4h''(\bar{x})}[1 - e^{-\frac{h''(\bar{x})}{D-h(\bar{x})}\{(z-D_0)^2 - 2(D-D_0)\cdot(z-D_0) + (D-D_0)\cdot[h(\bar{x})-D_0]\}}] \right\} \end{cases}$$

with

$$\lim_{\hat{x}\to\bar{x}} \frac{h'(x)}{h'(\hat{x})} = e^{\frac{1}{2}\frac{h''(\bar{x})}{D-h(\bar{x})}\{(z-D_0)^2 - 2(D-D_0)\cdot(z-D_0) + (D-D_0)\cdot[h(\bar{x})-D_0]\}}$$

If $h'(\bar{x})=0$ and $h''(\bar{x})=0$, then $\alpha_2(\bar{x},\hat{z})=\dfrac{(D-H)^2}{[D-h(\hat{x})]^2}$, $\beta_2(\bar{x},\hat{z})=0$ and

$$\gamma_2(\bar{x},\hat{z})=\kappa^2(\bar{x},z)+\frac{1}{4}\cdot\frac{(z-D_0)^2-2(D-D_0)\cdot(z-D_0)+(D-D_0)\cdot[h(\bar{x})-D_0]}{D-h(\bar{x})}\cdot h^{(4)}(\hat{x})$$

If $h'(x)\neq 0$, then

$$\begin{cases}\alpha_3(\hat{x},\hat{z})=\dfrac{[D-h(x^*)]^2[h'(x)]^2[h(\hat{x})-D_0]^2}{[D-h(x)]^2[h'(\hat{x})]^2[h(x^*)-D_0]^2}\\ \beta_3(\hat{x},\hat{z})=0\\ \gamma_3(\hat{x},\hat{z})=\dfrac{h''(\hat{x})}{h(\hat{x})-D_0}-\dfrac{3}{4}\cdot\dfrac{h'(\hat{x})^2}{[h(\hat{x})-D_0]^2}\\ \quad+\dfrac{1}{4}\cdot\dfrac{h''(\hat{x})^2}{h'(\hat{x})^2}-\dfrac{1}{2}\cdot\dfrac{h'''(\hat{x})}{h(\hat{x})-D_0}\\ \quad+\dfrac{D-D_0}{[D-h(x^*)]\cdot[h(x^*)-D_0]}\\ \quad\times\left\{\dfrac{3}{4}\cdot[h'(x^*)]^2\right.\\ \quad\cdot\dfrac{D-2\cdot h(x^*)+D_0}{[h(x^*)-D_0]\cdot[D-h(x^*)]}\\ \quad\left.-\dfrac{h''(x^*)}{1+h'(x^*)^2}\right\}\\ \quad\times\left\{\dfrac{h'(x^*)\cdot[h(\hat{x})-D_0]}{\{1+h'(x^*)^2\}\cdot h'(\hat{x})\cdot[h(x^*)-D_0]}\right\}^2\\ \quad+\left\{\kappa^2(x,z)+\dfrac{2\cdot h'(x)\cdot h'''(x)-h''(x)^2}{4\cdot h'(x)^2}\right.\\ \quad+\dfrac{1}{2}\cdot\dfrac{h''(x)}{D-h(x)}\\ \quad\left.+\dfrac{2\cdot h''(x)\cdot[D-h(x)]+3\cdot h'(x)^2}{4\cdot[D-h(x)]^2}\right\}\\ \quad\times\left\{\dfrac{h'(x)\cdot[h(\hat{x})-D_0]\cdot[D-h(x^*)]}{h'(\hat{x})\cdot[h(x^*)-D_0]\cdot[D-h(x)]}\right\}^2\end{cases}$$

If $h'(\bar{x})=0$ and $h''(\bar{x})\neq 0$, then

$$\begin{cases}\alpha_3(\bar{x},\hat{z})=e^{-(D-D_0)h''(\bar{x})}\\ \beta_3(\bar{x},\hat{z})=0\\ \gamma_3(\bar{x},\hat{z})=\dfrac{3}{4}\cdot\dfrac{h''(\bar{x})}{[h(\bar{x})-D_0]}\\ \quad+\left\{\kappa^2(\bar{x},z)+\dfrac{3}{4}\cdot\dfrac{h''(\bar{x})}{D-h(\bar{x})}\right\}\\ \quad\cdot e^{-(D-D_0)h''(\bar{x})}+\dfrac{h''(\bar{x})}{[h(\bar{x})-D_0]\cdot[D-h(\bar{x})]}\\ \quad\times\left\{\dfrac{[h(\bar{x})-D_0]}{4}-\dfrac{3}{4}(D-D_0)\right\}\\ \quad+\dfrac{1}{4}\cdot\dfrac{h^{(4)}(\bar{x})}{h''(\bar{x})}\cdot\{e^{-(D-D_0)h''(\bar{x})}-1\}\end{cases}$$

If $h'(\bar{x})=0$ and $h''(\bar{x})=0$, then $\alpha_2(\bar{x},\hat{z})=1$, $\beta_3(\bar{x},\hat{z})=0$, $\gamma_3(\bar{x},\hat{z})=\kappa^2(\bar{x},z)-\frac{1}{4}\cdot(D-D_0)\cdot h^{(4)}(\bar{x})$.

Appendix 2

When $h_1'(\bar{x})=0$, we define $P(\bar{x})=\lim_{x\to\bar{x}}P(\breve{x})$, $M(\bar{x})=\lim_{x\to\bar{x}}M(\hat{x})$, and $\dfrac{M(\bar{x})P_{\breve{x}\breve{x}}}{P}=\lim_{x\to\bar{x}}\dfrac{M(\hat{x})P_{\breve{x}\breve{x}}}{P}$. The formulas for $P(\breve{x})$, $M(\hat{x})$, $M(\hat{x})\dfrac{P_{\breve{x}\breve{x}}}{P}$ are given as follows:

If $h_1'(x)\neq 0$, then

$$P(\breve{x})=\sqrt{\frac{1}{\dfrac{\mathrm{d}\hat{x}}{\mathrm{d}\breve{x}}}}=\left[\frac{h_1'(\breve{x})}{h_1'(\hat{x})}\right]^{\frac{1}{2}}\left[\frac{D_0-h_1(x^*)}{D_0-h_1(\breve{x})}\right]^{\frac{1}{2}}\cdot\left[\frac{h_1(\hat{x})}{h_1(x^*)}\right]^{\frac{1}{2}}$$

If $h_1'(\bar{x})=0$, then $P(\bar{x})=e^{-\frac{D_0}{4}h_1''(\bar{x})}$.

If $h_1'(x)\neq 0$, then

$$M(\hat{x})=P(\breve{x})^4=\left[\frac{h_1'(\breve{x})}{h_1'(\hat{x})}\right]^2\left[\frac{D_0-h_1(x^*)}{D_0-h_1(\breve{x})}\right]^2\cdot\left[\frac{h_1(\hat{x})}{h_1(x^*)}\right]^2$$

If $h_1'(\bar{x})=0$, then $M(\bar{x})=e^{-D_0h_1''(\bar{x})}$.

If $h_1'(x)\neq 0$, then

$$P_{\breve{x}}=\left\{h_1'(\hat{x})^2-h_1''(\hat{x})h_1(\hat{x})+\frac{h_1'(\breve{x})^2h_1(\hat{x})^2[D_0-h_1(x^*)]}{[D_0-h_1(\breve{x})]^2h_1(x^*)}+\frac{h_1''(\breve{x})^2h_1(\hat{x})^2[D_0-h_1(x^*)]}{[D_0-h_1(\breve{x})]h_1(x^*)}+\frac{D_0h_1'(x^*)^2h_1(\hat{x})}{[1+h_1'(x^*)^2][D_0-h_1(x^*)]h_1(x^*)}\right\}\Bigg/\left\{2h_1(\hat{x})^2[h_1'(\hat{x})h_1'(\breve{x})]^{\frac{1}{2}}\cdot\left[\frac{h_1(\hat{x})}{D_0-h_1(\breve{x})}\right]^{\frac{1}{2}}\left[\frac{D_0-h_1(x^*)}{h_1(x^*)}\right]^{\frac{1}{2}}\right\}$$

If $h_1'(\bar{x})=0$ and $h_1''(\bar{x})\neq 0$, then

$$P_{\breve{x}}(\bar{x}) = \frac{h_1'''(\bar{x})}{2h_1''(\bar{x})}[e^{-\frac{1}{4}D_0 h_1''(\bar{x})} - e^{\frac{1}{4}D_0 h_1''(\bar{x})}]$$

If $h_1'(\bar{x}) = 0$ and $h_1''(\bar{x}) = 0$, then $P_{\breve{x}}(\bar{x}) = -\frac{1}{4}D_0 h_1'''(\bar{x})$, and

$$\begin{aligned} M(\hat{x})\frac{P_{\breve{x}\breve{x}}}{P} &= \frac{h_1''(\hat{x})}{h_1(\hat{x})} - \frac{3}{4}\cdot\frac{h_1'(\hat{x})^2}{h_1(\hat{x})^2} \\ &+ \frac{1}{4}\cdot\frac{h_1''(\hat{x})^2}{h_1'(\hat{x})^2} - \frac{1}{2}\cdot\frac{h_1'''(\hat{x})}{h_1(\hat{x})} \\ &+ \frac{D_0}{(D_0 - h_1(x^*))\cdot h_1(x^*)} \\ &\times\left\{\frac{3}{4}\cdot h_1'(x^*)^2\cdot\frac{D_0 - 2\cdot h_1(x^*)}{h_1(x^*)\cdot(D_0 - h_1(x^*))}\right. \\ &\left. - \frac{h_1''(x^*)}{1 + h_1'(x^*)^2}\right\} \\ &\times\left\{\frac{h_1'(x^*)\cdot h_1(\hat{x})}{\{1 + h_1'(x^*)^2\}\cdot h_1'(\hat{x})\cdot h_1(x^*)}\right\}^2 \\ &+ \left\{\frac{2\cdot h_1'(\breve{x})\cdot h_1'''(\breve{x}) - h_1''(\breve{x})^2}{4\cdot h_1'(\breve{x})^2}\right. \\ &+ \frac{1}{2}\cdot\frac{h_1''(\breve{x})}{D_0 - h_1(\breve{x})} \\ &\left. + \frac{2\cdot h_1''(\breve{x})\cdot(D_0 - h_1(\breve{x})) + 3\cdot h_1'(\breve{x})^2}{4\cdot\{D_0 - h_1(\breve{x})\}^2}\right\} \\ &\times\left\{\frac{h_1'(\breve{x})\cdot h_1(\hat{x})\cdot(D_0 - h_1(x^*))}{h_1'(\hat{x})\cdot h_1(x^*)\cdot(D_0 - h_1(\breve{x}))}\right\}^2 \end{aligned}$$

If $h_1'(\bar{x}) = 0$ and $h_1''(\bar{x}) \neq 0$, then

$$\begin{aligned} M(\bar{x})\frac{P_{\breve{x}\breve{x}}}{P} &= \frac{3}{4}\cdot\frac{h_1''(\bar{x})}{h_1(\bar{x})} \\ &+ \frac{3}{4}\cdot\frac{h_1''(\bar{x})}{D_0 - h_1(\bar{x})}\cdot e^{-D_0 h_1''(\bar{x})} \\ &+ \frac{h_1''(\bar{x})}{h_1(\bar{x})\cdot(D_0 - h_1(\bar{x}))} \\ &\times\left\{\frac{h_1(\bar{x})}{4} - \frac{3}{4}D_0\right\} \\ &+ \frac{1}{4}\cdot\frac{h_1^{(4)}(\bar{x})}{h_1''(\bar{x})}\cdot\{e^{-D_0 h_1''(\bar{x})} - 1\} \end{aligned}$$

If $h_1'(\bar{x}) = 0$ and $h_1''(\bar{x}) = 0$, then

$$M(\bar{x})\frac{P_{\breve{x}\breve{x}}}{P} = -\frac{1}{4}\cdot D_0\cdot h_1^{(4)}(\bar{x})$$

References

1 Jensen FB, Kuperman WA, Porter MB, et al. Computational Ocean Acoustics. New York: American Institute of Physics, 1994

2 Fishman L. One-way wave propagation methods in direct and inverse scalar wave propagation modeling. Radio Sci, 1993, 28(5): 865—876

3 Fishman L, Gautesen AK and Sun AK. Uniform high-frequency approximations of the square root Helmholtz operator symbol. Wave Motion, 1997, 26(2): 127—161

4 Lu YY and McLaughlin JR. The Riccati method for the Helmholtz equation. J Acoust Soc Am, 1996, 100(3): 1432—1446

5 Tappert FD. The parabolic approximation method. In: Wave Propagation and Underwater Acoustics. Berlin, New York: Springer-Verlag, 1977

6 Lee D and Pierce AD. Parabolic equation development in recent decade. J Comput Acoust, 1995, 3: 95—173

7 Lu YY. One-way large range step methods for Helmholtz waveguides. J Comput Phys, 1999, 152: 231—250

8 Abrahamsson L and Kreiss HO. Numerical solution of the coupled mode equations in duct acoustic. J Comput Phys, 1994, 111: 1—14

9 Larsson E and Abrahamsson L. Parabolic wave equation versus the Helmholtz equation in Ocean acoustics. In: Mathematics and Numerical Aspects of Wave Propagation, SIAM, Philadelphia, 1998, 582—584

10 Tessmer E, Kosloff D and Bethe A. Elastic wave propagation simulation in the presence of surface topography. Geophys J Int, 1992, 108: 621—632

11 Dougalis VA and Kampanis NA. Finite element methods for the parabolic equation with interfaces. J Comput Acoust, 1996, 4: 55—88

12 Evans R. The flattened surface parabolic equation. J Acoust Soc Am, 1998, 104: 2167—2173

13 Lu YY, Huang J and McLauphilin JR. Local orthogonal transformation and one-way methods for acoustics waveguides. Wave Motion, 2001, 34: 193—207

14 Zhu JX and Lu YY. Large range step method for acoustic waveguide with two layer media. Progress in Natural Science, 2002, 12: 820—825